and 4384, the phase μ in the simple Rydberg formula being about 0·02, and the limit being approximately that of the Diffuse Series of Parhelium, but the measurements of the wave-lengths of the lines are not sufficiently accurate to establish fully their series relations. A number of other lines have been observed, but an investigation with higher dispersion will be required for a full elucidation of the spectrum.]

Periodic Irrotational Waves of Finite Height.

By T. H. Havelock, F.R.S.

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1. The method of Stokes for waves of finite height on deep water consists in working upwards by successive steps from the infinitesimal wave towards the highest possible wave. Although lacking formal proof of convergence, it is generally accepted that the method is valid, but that it does not include the highest possible wave when the crests form wedges of 120°.

For the highest wave itself we have Michell's analysis by a distinct method, also involving an infinite series whose convergence has to be assumed.*

The theoretical position of Stokes' method has been stated concisely by Prof. Burnside in a recent criticism†:—

"The complete result would be to express the co-ordinates x and y in terms of ϕ and ψ in the form

$$x = -\phi + be^{\psi}\sin\phi + \sum_{2}^{\infty} b^n \mathrm{P}_n(b)\, e^{n\psi}\sin n\phi,$$

$$y = -\psi + be^{\psi}\cos\phi + \sum_{2}^{\infty} b^n \mathrm{Q}_n(b)\, e^{n\psi}\cos n\phi, \tag{1}$$

where $\mathrm{P}_n(b)$, $\mathrm{Q}_n(b)$ are power series in b.

"These results have a meaning and can be used for actual approximate calculation only, if P_n, Q_n are convergent power series when b does not exceed some value, say b_0, while for suitable values of b and for real negative values of ψ, the series for x and y are convergent.

"Until the form of the power series P_n and Q_n have been determined, it is impossible to deal with their convergence. Assuming that they are convergent, it is clear from physical considerations that there must be an upper limit b' for b in order that the series for x and y may be convergent for negative values of ψ, and there are no means of determining b'."

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* J. H. Michell, 'Phil. Mag.,' Ser. 5, vol. 36, p. 430 (1893).

† W. Burnside, 'Lond. Math. Soc. Proc.,' Ser. 2, vol. 15, p. 26 (1916).

Prof. Burnside concludes that Stokes' method cannot be used for numerical calculation as it is not known whether the corresponding value of b is less than the above value b'.

In the following notes a general method is suggested, which includes waves of all possible heights, ranging from the highest wave down to the simple infinitesimal wave. The method consists of a simple and direct extension of Michell's analysis for the highest wave. The advantage is a theoretical one which may be expressed in this form: the parameter does not have, as in Stokes' series, an undetermined upper limit, but it enters in the form $e^{-2\alpha}$, where α may have any positive value, including zero.

It should be stated that here, again, we have infinite processes for which no formal proof of convergence is given: we have to rely meantime upon a numerical study of the series. However, in addition, we can compare the method with that of Stokes for waves short of the highest; in this case numerical results obtained by the two methods are the same, as might be expected.

Extending this comparison to the highest possible wave, we get a value for the quantity b' referred to previously, that is, the value of the parameter for which Stokes' series for the elevation become divergent. We obtain b' as $\frac{1}{3}-b_1$, where b_1 has the value 0·0414 approximately, or we have $b' = 0·291...$, the value for b_1 being slightly less than the true value.

The discussion is arranged in the following order: Michell's form for the highest wave, its generalisation by means of the surface condition, method of approximation for the coefficients, calculation for the highest wave, the values when $e^{-2\alpha} = \frac{3}{4}$, comparison with Stokes' series, determination of b', further numerical examples and remarks upon the values of the coefficients.

2. It was shown by Stokes that the highest possible wave, under constant pressure at the free surface, has crests in the form of wedges of 120°. It follows directly from his argument, as a simple extension, that the crests will meet at the same angle for the highest possible wave under any assigned surface pressure provided the pressure is stationary in value over the crests.

Consider any assigned surface pressure of this character which is finite, continuous and periodic. To determine the form of the highest possible periodic wave, we may follow Michell's analysis for the case of constant pressure up to the stage at which the coefficients are determined from the given surface condition.

We might then begin with the form given in (5) below, but we may recall briefly Michell's argument. Take Ox horizontal, Oy vertical and downwards, with O at a crest. Let the successive crests be given by $\phi = \pm n\pi$, n integral; and let the upper surface be $\psi = 0$. If θ be the inclination of the wave line to the horizontal at a point ϕ, assume

$$\frac{\partial\theta}{\partial\phi} = a_0 + a_1 \cos 2\phi + a_2 \cos 4\phi + \dots. \tag{2}$$

This is equivalent to assuming that under the given conditions for the surface pressure, the ratio of the curvature of the wave line to the velocity is finite and continuous throughout a wave-length; in that case $\partial\theta/\partial\phi$ can be expanded in a uniformly convergent Fourier series. In the numerical calculations which are needed later, the practical success of the method of approximation depends upon $a_1, a_2, \dots$, being small compared with a_0, and, in fact, upon the series converging rapidly.

With the notation, $w = \phi + i\psi$, $z = x + iy$, $\mathrm{U} = \log(dz/dw)$, Michell showed that

$$\frac{d\mathrm{U}}{dw} - i\,(a_0 + a_1 e^{2iw} + a_2 e^{4iw} + \dots), \tag{3}$$

is a function which is real over the surface $\psi = 0$, and possesses only simple poles, which are at the wave crests.

Suppose that near a crest, say $w = 0$, we have $dz/dw = \mathrm{A}w^n$, then $q^2 = \text{const.} \times r^{-2n/(n+1)}$, where q is fluid velocity and r is distance from the summit. But, since the pressure is constant in the neighbourhood of the crests, we have $q^2 = 2gy$, and hence $n = -\frac{1}{3}$. It follows that the function (3) differs by only a constant from the quantity $-\frac{1}{3}\Sigma\,(w - n\pi)^{-1}$. Hence, after adjusting the constants and integrating, we find for dz/dw the form

$$\frac{dz}{dw} = (-i \sin w)^{-1/3} e^{-iw/3} (1 + c_1 e^{2iw} + c_2 e^{4iw} + \dots), \tag{4}$$

the real root of $(-i \sin w)^{-1/3}$ being taken along $\phi = 0$. The units are such that the wave velocity V, or the velocity at $\psi = \infty$, and the wave-length L are given by

$$\mathrm{V} = 2^{-1/3}; \qquad \mathrm{L} = \pi/\mathrm{V} = 2^{1/3}\pi.$$

It is convenient to invert (4) and write

$$\frac{dw}{dz} = (-i \sin w)^{1/3} e^{iw/3} (1 + b_1 e^{2iw} + b_2 e^{4iw} + b_3 e^{6iw} + \dots). \tag{5}$$

3. The coefficients $b_1, b_2, \dots$, are now to be determined by the pressure condition at the free surface. So far, we have stipulated only that the pressure at $\psi = 0$ shall be finite, continuous, periodic, and stationary at the points $\phi = n\pi$. For our present purpose we shall leave this pressure distribution undetermined, except for these conditions. We shall assume

that it is possible for some assigned stream-line below the surface, say the line $\psi = \alpha$, to be a line of constant pressure. Thus we shall determine $b_1, b_2, \ldots$, directly by applying the condition for constant pressure to (5) when $\psi = \alpha$. The surface $\psi = \alpha$ will then be a possible free wave surface, and free waves will be given by assigning any value to α in the range zero to infinity; thus, by working down from the highest possible wave, we include in one scheme free waves of any permissible height.

The condition that the pressure is constant for $\psi = \alpha$ is

$$q^2 = 2gy + \text{constant}. \tag{6}$$

It is convenient to use an equivalent form obtained by differentiating (6) with respect to ϕ, namely

$$\frac{\partial q^4}{\partial \phi} = 4g\frac{\partial \phi}{\partial y}. \tag{7}$$

This has to be used when

$$\frac{dw}{dz} = \{-i\sin(\phi + i\alpha)\}^{1/3} e^{i(\phi+i\alpha)/3}(1 + \beta_1 e^{2i\phi} + \beta_2 e^{4i\phi} + \ldots), \tag{8}$$

where $\beta_1 = b_1 e^{-2\alpha}, \qquad \beta_2 = b_2 e^{-4\alpha}, \ldots.$

Multiplying by the conjugate complex and squaring, we obtain

$$q^4 = e^{-4\alpha/3}(\sinh^2\alpha + \sin^2\phi)^{2/3}(\mathrm{D}_0 + 2\mathrm{D}_1\cos 2\phi + 2\mathrm{D}_2\cos 4\phi + \ldots), \tag{9}$$

where

$$\mathrm{D}_0 = 1 + 4\beta_1{}^2 + (2\beta_2 + \beta_1{}^2)^2 + (2\beta_3 + 2\beta_1\beta_2)^2 + \ldots,$$

$$\mathrm{D}_1 = 2\beta_1 + 2\beta_1(2\beta_2 + \beta_1{}^2) + (2\beta_2 + \beta_1{}^2)(2\beta_3 + 2\beta_1\beta_2) + \ldots,$$

$$\mathrm{D}_2 = 2\beta_2 + \beta_1{}^2 + 2\beta_1(2\beta_3 + 2\beta_1\beta_2) + (2\beta_2 + \beta_1{}^2)(2\beta_4 + \beta_2{}^2 + 2\beta_1\beta_3) + \ldots,$$

$$\mathrm{D}_3 = 2\beta_3 + 2\beta_1\beta_2 + 2\beta_1(2\beta_4 + \beta_2{}^2 + 2\beta_1\beta_3) + \ldots,$$

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Differentiating (9) with respect to ϕ, we can take out a factor $(\sinh^2\alpha + \sin^2\phi)^{-1/3}$, and can collect the other terms into a sine series in even multiples of ϕ. However, we take out also the common factor $\sin\phi$, because we then have $\partial q^4/\partial\phi$ in a form which reduces directly to the proper form for the highest wave ($\alpha = 0$), and, in addition, we find that the numerical calculations converge more rapidly. After some reduction, we obtain in this way

$$\frac{\partial q^4}{\partial \phi} = \tfrac{4}{3}e^{-4\alpha/3}\sin\phi(\sinh^2\alpha + \sin^2\phi)^{-1/3}(\mathrm{A}_1\cos\phi + \mathrm{A}_3\cos 3\phi + \ldots), \tag{10}$$

where

$$\begin{aligned} A_1 &= D_0+\tfrac{5}{2}D_1+6D_2+3(3D_3+4D_4+\ldots)-3(D_1+2D_2+3D_3+\ldots)\cosh 2\alpha, \\ A_3 &= \tfrac{5}{2}D_1+4D_2+3(3D_3+4D_4+\ldots)-3(2D_2+3D_3+\ldots)\cosh 2\alpha, \\ A_5 &= 4D_2+\tfrac{11}{2}D_3+3(4D_4+5D_5+\ldots)-3(3D_3+4D_4+\ldots)\cosh 2\alpha, \\ A_7 &= \tfrac{11}{2}D_3+7D_4+3(5D_5+\ldots)-3(4D_4+\ldots)\cosh 2\alpha, \end{aligned}$$

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For the other side of equation (7), using I to denote the imaginary part Q of a complex quantity $P+iQ$, we have

$$\frac{\partial\phi}{\partial y} = -\mathrm{I}\,e^{-\alpha/3}(\sinh^2\alpha+\sin^2\phi)^{1/6}e^{-i(\theta-\phi)/3}\sum\beta_r e^{2ri\phi}, \tag{11}$$

where $$\mathrm{R}e^{i\theta} = \cos\phi\sinh\alpha - i\sin\phi\cosh\alpha.$$

To expand this in a form similar to (10), we notice that

$$e^{-i(\theta-\phi)/3} = (1-e^{-2\alpha}e^{2i\phi})^{1/6}(1-e^{-2\alpha}e^{-2i\phi})^{-1/6},$$

$$(\sinh^2\alpha+\sin^2\phi)^{1/2} = \tfrac{1}{2}e^{\alpha}(1-e^{-2\alpha}e^{2i\phi})^{1/2}(1-e^{-2\alpha}e^{-2i\phi})^{1/2}.$$

Hence we have

$$\frac{\partial\phi}{\partial y} = -\mathrm{I}\,\tfrac{1}{2}e^{2\alpha/3}(\sinh^2\alpha+\sin^2\phi)^{-1/3}\sum_{r=0}^{\infty}\beta_r(1-e^{-2\alpha}e^{2i\phi})^{2/3}(1-e^{-2\alpha}e^{-2i\phi})^{1/3}e^{2ri\phi}. \tag{12}$$

We now expand the two binomial factors after the sign of summation in series valid for the whole range of ϕ and for all positive values of α. We can then write down the coefficient of $e^{2ni\phi}$, and so obtain $\partial\phi/\partial y$, involving a series of sines of even multiples of ϕ; as before, we take out a common factor $\sin\phi$, and obtain the result

$$\frac{\partial\phi}{\partial y} = \tfrac{1}{3}e^{-4\alpha/3}\sin\phi(\sinh^2\alpha+\sin^2\phi)^{-1/3}(B_1\cos\phi+B_3\cos 3\phi+\ldots), \tag{13}$$

where the B's are linear functions of the β's, with coefficients which are functions of $e^{-2\alpha}$. In practice, these can be obtained directly from (12) to any required degree of approximation; general expressions can be put in the following exact forms

$$B_{2n+1} = B_{0,2n+1}+\beta_1 B_{1,2n+1}+\ldots+\beta_r B_{r,2n+1}+\ldots,$$

$$B_{r,2n+1} = \sum_{s=n+r+1}^{\infty}C_{-2s}-\sum_{s=n-r+1}^{\infty}C_{2s},$$

$$\begin{aligned} C_{2s} &= 3\,\frac{-\frac{2}{3}(-\frac{2}{3}+1)\ldots(-\frac{2}{3}+s-1)}{s\,!}\,e^{-2(s-1)\alpha}F(-\tfrac{1}{3},-\tfrac{2}{3}+s,s+1,e^{-4\alpha}), \\ C_{-2s} &= 3\,\frac{-\frac{1}{3}(-\frac{1}{3}+1)\ldots(-\frac{1}{3}+s-1)}{s\,!}\,e^{-2(s-1)\alpha}F(-\tfrac{1}{3}+s,-\tfrac{2}{3},s+1,e^{-4\alpha}), \\ C_0 &= 3e^{2\alpha}F(-\tfrac{1}{3},-\tfrac{2}{3},1,e^{-4\alpha}), \end{aligned} \tag{14}$$

where F represents the hypergeometric series.

We can now apply the surface condition (7) by substituting the expansions (10) and (13) and equating coefficients of $\cos\phi$, $\cos 3\phi$, ...

We obtain, as in Stokes' method, an infinite series of equations of the form $A_{2n+1} = gB_{2n+1}$, from which the quantities g, β_1, β_2, β_3, ..., are to be obtained in practice by successive approximation from the equations taken in order.

Up to terms of the third order the equations are

$$\begin{aligned}
&1+5\beta_1+12\beta_2+10\beta_1^2+28\beta_1\beta_2+5\beta_1^3+18\beta_3 \\
&\qquad\qquad -6(\beta_1+2\beta_2+\beta_1^2+5\beta_1\beta_2+\beta_1^3+3\beta_3)\cosh 2\alpha \\
&\qquad = g(B_{01}+\beta_1 B_{11}+\beta_2 B_{21}+\beta_3 B_{31}), \\
&5\beta_1+8\beta_2+4\beta_1^2+28\beta_1\beta_2+5\beta_1^3+18\beta_3-6(2\beta_2+\beta_1^2+3\beta_1\beta_2+3\beta_3)\cosh 2\alpha \\
&\qquad = g(B_{03}+\beta_1 B_{13}+\beta_2 B_{23}+\beta_3 B_{33}), \\
&8\beta_2+4\beta_1^2+11\beta_1\beta_2+11\beta_3-18(\beta_1\beta_2+\beta_3)\cosh 2\alpha \\
&\qquad = g(B_{05}+\beta_1 B_{15}+\beta_2 B_{25}+\beta_3 B_{35}), \\
&11\beta_3+11\beta_1\beta_2 = g(B_{07}+\beta_1 B_{17}+\beta_2 B_{27}+\beta_3 B_{37}).
\end{aligned} \tag{15}$$

It might appear, from the quantity $\cosh 2\alpha$ on the left, and from the factor $e^{2\alpha}$ in the expressions for the B's in (14), that there are terms in these equations which become infinitely large as α increases indefinitely. But we have $\beta_1 = b_1 e^{-2\alpha}$, $\beta_2 = b_2 e^{-4\alpha}$, $\beta_3 = b_3 e^{-6\alpha}$, ..., therefore, if we write the equations (15) as a set of equations for the coefficients b_1, b_2, ..., this difficulty disappears. In this connection we may recall the initial assumption that the series in (5), namely, $1+b_1 e^{2iw}+b_2 e^{4iw}+\dots$, is absolutely convergent, otherwise the analysis has no meaning.

The infinite set of equations, given to the third order in (15), has to be treated by Stokes' method; that is, assuming the process to be convergent, the equations taken in succession yield approximations to g, β_1, β_2, ..., for any assigned numerical value of α. But there is a difference between these equations and the corresponding set in Stokes' analysis. In the latter, the first coefficient, say b, is arbitrary, and the successive equations have their lowest terms of order zero, one, two, and so on, respectively; thus g and the remaining coefficients are found as power series in b. But in (15), we have a term independent of the β's on the right-hand side of each equation; thus the solution, if practicable, leads to a set of numerical values of g, β_1, β_2,... for a definite numerical value of α. We may notice, in passing, that for a first rough approximation $gB_{01} = 1$; and as B_{01} does not differ much from unity for any value of α, the coefficients β_1, β_2, ..., are of the order of magnitude of B_{03}, B_{05}, ..., respectively.

4. The method of approximation used in the following calculations may be described by considering first the simplest form of the equations, namely,

when $\alpha = 0$. The hypergeometric series in (14) can be summed in this case and we find

$$\mathrm{B}_{r,2n+1} = \frac{18\sqrt{3}}{\pi}\,\frac{6r+1}{9(2n+1)^2-(6r+1)^2}. \tag{16}$$

The equations (15) reduce to

$$\begin{aligned} 1-\beta_1+4\beta_1^2-2\beta_1\beta_2-\beta_1^3 &= k\left(\tfrac{1}{8}-\tfrac{7}{40}\beta_1-\tfrac{13}{160}\beta_2-\tfrac{19}{352}\beta_3\right), \\ 5\beta_1-4\beta_2-2\beta_1^2+10\beta_1\beta_2+5\beta_1^3 &= k\left(\tfrac{1}{80}+\tfrac{7}{32}\beta_1-\tfrac{13}{88}\beta_2-\tfrac{19}{280}\beta_3\right), \\ 8\beta_2+4\beta_1^2-7\beta_1\beta_2-7\beta_3 &= k\left(\tfrac{1}{224}+\tfrac{7}{196}\beta_1+\tfrac{13}{56}\beta_2-\tfrac{19}{136}\beta_3\right), \\ 11\beta_1\beta_2+11\beta_3 &= k\left(\tfrac{1}{440}+\tfrac{7}{392}\beta_1+\tfrac{13}{272}\beta_2+\tfrac{19}{86}\beta_3\right), \end{aligned} \tag{17}$$

with $k = 18g\sqrt{3}/\pi$.

These are Michell's equations for the highest wave. Without specifying any definite method of approximation, Michell states that sufficiently close values are given by

$$g = 0{\cdot}832, \qquad \beta_1 = 0{\cdot}0397, \qquad \beta_2 = 0{\cdot}0094, \qquad \beta_3 = 0{\cdot}002. \tag{18}$$

In order to compare results for different values of α, it is desirable to adopt some consistent scheme of approximation.

In general, in the equations (15), we substitute

$$\begin{aligned} g\mathrm{B}_{01} &= 1+k_1\beta_1+k_2\beta_1^2+k_3\beta_1^3+\ldots, \\ \beta_2 &= b_2\beta_1^2+b_3\beta_1^3+\ldots, \\ \beta_3 &= c_3\beta_1^3+c_4\beta_1^4+\ldots, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{19}$$

For a first approximation, write down the first two equations up to terms in β_1, and we get two equations from which to determine k_1 and β_1. The first of these equations is, in fact, independent of β_1 on account of the form of (19).

For a second approximation, retain the value of k_1 so determined, and write down the first three equations of (15) up to the terms in β_1^2, the first of the three being again independent of β_1; from these, we determine k_2, β_2, and a second approximation to β_1. For the third stage, using the values of k_1, k_2, and b_2 already found, and writing down the first four equations of (15) up to the terms in β_1^3, we determine k_3, b_3, c_3, and a third approximation to β_1. Using (19) we obtain the corresponding values of g, β_2, β_3, ..., at any stage. The nth approximation to β_1 is given by an equation of the nth degree in β_1; but there is no difficulty in practice as to the particular root since we follow it through from the first approximation.

The method is simple in plan, if somewhat tedious in practice; so it is not necessary to give the details of the following calculations.

Taking the particular case (17), we may write down one set of equations to illustrate the type. After the first two stages, we obtain $k_1 = 0{\cdot}4$, $k_2 = 7{\cdot}6$, $b_2 = 4{\cdot}67$: using these values we find for the next stage the equations

$$\left.\begin{aligned} k_3 - 0{\cdot}65 b_3 - 0{\cdot}432 c_3 &= 1{\cdot}18 \\ \beta_1^3 (0{\cdot}1 k_3 + 2{\cdot}82 b_3 - 0{\cdot}543 c_3 - 44{\cdot}25) + 17{\cdot}745 \beta_1^2 - 3{\cdot}21 \beta_1 + 0{\cdot}1 &= 0 \\ \beta_1^3 (0{\cdot}036 k_3 - 6{\cdot}14 b_3 + 5{\cdot}88 c_3 + 41{\cdot}68) - 34{\cdot}68 \beta_1^2 + 0{\cdot}332 \beta_1 - 0{\cdot}036 &= 0 \\ \beta_1^3 (0{\cdot}018 k_3 + 0{\cdot}38 b_3 - 9{\cdot}1 c_3 - 53{\cdot}77) + 2{\cdot}15 \beta_1^2 + 0{\cdot}15 \beta_1 + 0{\cdot}018 &= 0 \end{aligned}\right\} \tag{20}$$

Eliminating k_3, b_3, c_3, we get a cubic for β_1, of which the required root is 0·0407, the previous stages having given the values 0·0311, 0·039. Also from (20) we find $k_3 = 30$, $b_3 = 35$, $c_3 = 40$. Collecting the results to this stage, we find

$$g = 0{\cdot}833, \qquad \beta_1 = 0{\cdot}0407, \qquad \beta_2 = 0{\cdot}0106, \qquad \beta_3 = 0{\cdot}0027. \tag{21}$$

These values are rather higher than those given by Michell (18). In order to determine β_1 more closely, the approximation has been carried to the fourth stage, with the result

$$g = 0{\cdot}833, \quad \beta_1 = 0{\cdot}0414, \quad \beta_2 = 0{\cdot}0114, \quad \beta_3 = 0{\cdot}0042, \quad \beta_4 = 0{\cdot}0014. \tag{22}$$

With these values, the ratio of h, the height of the wave, to L, the wavelength, is given by

$$\begin{aligned} h/\mathrm{L} &= (\text{velocity at trough})^2/2g\mathrm{L} \\ &= (1-\beta_1+\beta_2-\beta_3+\beta_4)^2/2^{4/3} g\pi = 0{\cdot}1418. \end{aligned} \tag{23}$$

An interesting point about the series $1+\beta_1 e^{2i\phi}+\beta_2 e^{4i\phi}+\ldots$ for the highest wave is the smallness of all the coefficients $\beta_1, \beta_2, \ldots$, compared with the first term, namely, unity; on the other hand, the numerical values obtained do not suggest a rapid convergence of the series after the first term. It appears, from the method of approximation, and from the fact that all the quantities $\mathrm{B}_{01}, \mathrm{B}_{03}, \ldots$, are positive, that successive approximations will increase the values of the coefficients. A test for the sum of the series, compared with the value of g, is obtained by considering the velocity near a crest.

Near $\phi = 0$, we have $dw/dz = \phi^{1/3} e^{-i\pi/6} (1+\beta_1+\beta_2+\ldots)$.

Therefore $q^2 = \phi^{2/3} (1+\beta_1+\beta_2+\ldots)^2$ and $z = \frac{3}{2}\phi^{2/3} e^{i\pi/6}/(1+\beta_1+\beta_2+\ldots)$; and since $q^2 = 2gy$, it follows that we should have $(1+\beta_1+\beta_2+\ldots)^3/g = 1{\cdot}5$. But with the values given in (22), this expression has the value 1·42. This is perhaps a severe test; a simpler criterion is to write down the successive convergents to any one coefficient; for example, those for the leading coefficient β_1 are 0·0311, 0·0390, 0·0407, and 0·0414.

5. Returning to the general equations (15), we consider a wave short of the highest and we select the case $e^{-2\alpha} = \frac{3}{4}$. We shall find that this corresponds to a value of about $\frac{1}{4}$ for Stokes' parameter b.

The coefficients $B_{r, 2n+1}$ have to be calculated from the relations (14); the hypergeometric series are, of course, convergent, and the values can be obtained to any required degree of accuracy. Substituting the numerical values in (15) we obtain the equations

$$\begin{aligned}
&1-1{\cdot}25\beta_1-0{\cdot}5\beta_2+3{\cdot}75\beta_1^2-3{\cdot}25\beta_1\beta_2-1{\cdot}25\beta_1^3-0{\cdot}75\beta_3 \\
&\qquad\qquad = g\,(1{\cdot}081-2{\cdot}584\beta_1-1{\cdot}53\beta_2-1{\cdot}281\beta_3), \\
&5\beta_1-4{\cdot}5\beta_2-2{\cdot}25\beta_1^2+9{\cdot}25\beta_1\beta_2+5\beta_1^3-0{\cdot}75\beta_3 \\
&\qquad\qquad = g\,(0{\cdot}0166+2{\cdot}133\beta_1-2{\cdot}322\beta_2-1{\cdot}429\beta_3), \\
&8\beta_2+4\beta_1^2-7{\cdot}75\beta_1\beta_2-7{\cdot}75\beta_3 = g\,(-0{\cdot}0157+0{\cdot}2767\beta_1+2{\cdot}237\beta_2-2{\cdot}274\beta_3) \\
&\qquad 11\beta_3+11\beta_1\beta_2 = g\,(-0{\cdot}0125+0{\cdot}0865\beta_1+0{\cdot}3243\beta_2+2{\cdot}254\beta_3).
\end{aligned} \tag{24}$$

We carry out now the successive approximations described in the previous section. At the third stage, we find

$$g = 0{\cdot}9246, \quad \beta_1 = 0{\cdot}00273, \quad \beta_2 = -0{\cdot}0034, \quad \beta_3 = -0{\cdot}0013. \tag{25}$$

Comparing these values with those for the highest wave given in (21), we see that the β's are much smaller; on the other hand, there may be greater difficulty in obtaining their values accurately, because of the later stage at which the β's begin to diminish steadily in absolute value. We shall find this impression confirmed later when we try smaller values of $e^{-2\alpha}$.

To find the ratio h/L for this wave, we have

$$(\text{velocity at crest})^2 = 2^{-2/3}(1-e^{-2\alpha})^{2/3}(1+\beta_1+\beta_2+\beta_3+\ldots)^2,$$

$$(\text{velocity at trough})^2 = 2^{-2/3}(1+e^{-2\alpha})^{2/3}(1-\beta_1+\beta_2-\beta_3+\ldots)^2.$$

Taking the difference, and dividing by $2g$, we find h; and since $L = 2^{1/3}\pi$, we have $h/L = 0{\cdot}0898$. Stokes' parameter b is, to a first approximation, $\pi h/L$; hence this wave corresponds to b equal to $\frac{1}{4}$ nearly.

6. We have now two methods for a wave of finite height, namely, that described above and Stokes' method. The two can be shown to be in agreement in any particular case.

From (8), we have, on the wave surface $\psi = \alpha$,

$$2^{-1/3}\frac{dz}{dw} = (1-e^{-2\alpha}e^{2i\phi})^{-1/3}(1+\beta_1 e^{2i\phi}+\beta_2 e^{4i\phi}+\ldots)^{-1}. \tag{26}$$

For any wave below the highest possible, that is provided α is not zero, the first factor on the right of (26) can be expanded in a series valid for all values of ϕ; hence, under these conditions, we have on the surface $\psi = \alpha$,

$$2^{-1/3}\frac{dz}{dw} = 1 + A_1 e^{2i\phi} + A_2 e^{4i\phi} + A_3 e^{6i\phi} + \ldots, \tag{27}$$

where

$$A_1 = \tfrac{1}{3} e^{-2\alpha} - \beta_1,$$

$$A_2 = \tfrac{2}{9} e^{-4\alpha} - \tfrac{1}{3}\beta_1 e^{-2\alpha} - (\beta_2 - \beta_1^2),$$

$$A_3 = \tfrac{14}{81} e^{-6\alpha} - \tfrac{2}{9}\beta_1 e^{-4\alpha} - \tfrac{1}{3}(\beta_2 - \beta_1^2) e^{-2\alpha} - (\beta_3 - 2\beta_1\beta_2 + \beta_1^3),$$

$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now Stokes' method gives z, and dz/dw, in the form of a series like (27); write this as

$$cz = 1 + C_1 e^{2i\phi} + C_2 e^{4i\phi} + C_3 e^{6i\phi} + \ldots. \tag{28}$$

From Stokes' equations, C_2, C_3, ..., are obtained as power series in C_1; these have been carried up to the tenth order by Wilton,* whose results we quote now—in so far as they are needed here—

$$-C_1 = b,$$

$$C_2 = b^2 + 0{\cdot}5b^4 + 2{\cdot}417b^6 + 15{\cdot}597b^8 + 64{\cdot}08b^{10},$$

$$-C_3 = 1{\cdot}5b^3 + 1{\cdot}583b^5 + 8{\cdot}215b^7 + 55{\cdot}01b^9,$$

$$C_4 = 2{\cdot}667b^4 + 4{\cdot}347b^6 + 24{\cdot}01b^8 + 166{\cdot}2b^{10},$$

$$-C_5 = 5{\cdot}208b^5 + 11{\cdot}53b^7 + 674b^9,$$

$$\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\frac{2\pi c^2}{g\lambda} = 1 + b^2 + 3{\cdot}5b^4 + 19{\cdot}08b^6 + 154{\cdot}73b^8 + 1297b^{10}. \tag{29}$$

With the units adopted here, the last expression corresponds to $1/g$. Further, in Stokes' investigations the wave-length was taken as 2π, while in the above work we have used π; the result is that in comparing the two methods by means of (27) and (29), C_1, C_2, C_3, C_4, ..., correspond respectively to A_1, $\frac{1}{2}A_2$, $\frac{1}{3}A_3$, $\frac{1}{4}A_4$, ...

For the numerical calculations in the case $e^{-2\alpha} = \frac{3}{4}$, we use the values of β_1, β_2, and β_3 given in (25); then from (27) we obtain

$$A_1 = 0{\cdot}24727; \quad \tfrac{1}{2}A_2 = 0{\cdot}06385; \quad \tfrac{1}{3}A_3 = 0{\cdot}0249;$$

$$\tfrac{1}{4}A_4 = 0{\cdot}0115; \quad \tfrac{1}{5}A_5 = 0{\cdot}0058; \ \ldots\ldots \tag{30}$$

On the other hand, if we take b equal to A_1, we get from this series in (29)

$$-C_1 = 0{\cdot}24727; \quad C_2 = 0{\cdot}06382; \quad -C_3 = 0{\cdot}0248;$$

$$C_4 = 0{\cdot}0114; \quad -C_5 = 0{\cdot}0058; \ \ldots\ldots \tag{31}$$

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* J. R. Wilton, 'Phil. Mag.,' Ser. 6, vol. 27, p. 385 (1914).

It is unnecessary to carry the calculation further to show the numerical agreement between the two methods for waves short of the highest. It may be noticed that in the above comparison we have gone up to the coefficient β_3 of the present method; to obtain the agreement shown above, we have had to use the Stokes' series as far as the tenth order in the parameter.

7. From the comparison between (27) and (28), we see that, for waves lower than the highest, we are in effect dealing with a Stokes' series whose parameter has the value $\frac{1}{3}e^{-2\alpha}-\beta_1$. If we applied Stokes' method directly to (27), we should obtain $A_2, A_3, \ldots$, in the usual way as power series in this parameter, and the quantity $e^{-2\alpha}$ would be a superfluous arbitrary parameter. On the other hand, the present method gives a definite value of β_1 for an assigned value of α, or theoretically gives a functional relation between β_1 and α. The method definitely connects a wave of any height with the highest possible wave, and any possible wave-form is given as one of a family whose limiting curve has crests consisting of wedges of 120°.

Consider the expansion from the form (26) to the corresponding Stokes' form (27) or (28). Assuming the convergence of the series with the β-coefficients, the expansion is valid over the whole range of ϕ for all positive values of α, excluding zero; it is also valid for α zero, with the exception of the points $\phi = n\pi$, n integral. In other words, the comparison confirms the view that Stokes' series for the elevation is valid throughout, with the exception of the actual crests of the highest possible wave.

We can now estimate the limiting value of the Stokes' parameter b for convergence at the crests. To do this, we compare the series (27) for the highest wave with a Stokes' series, for points other than the crests.

For the highest wave $\alpha = 0$, we found

$$\beta_1 = 0{\cdot}0414, \qquad \beta_2 = 0{\cdot}0114, \qquad \beta_3 = 0{\cdot}0042, \qquad \beta_4 = 0{\cdot}0014.$$

Hence the expansion should be a Stokes' series with the parameter $\frac{1}{3}-0{\cdot}0414$, or say $0{\cdot}2919$. Making the comparison between (27) and (29) with these values, we find

$$\begin{array}{lll} A_1 = 0{\cdot}2919\,; & \frac{1}{2}A_2 = 0{\cdot}0993\,; & \frac{1}{3}A_3 = 0{\cdot}0523\,; \\ -C_1 = 0{\cdot}2919\,; & C_2 = 0{\cdot}0914\,; & -C_3 = 0{\cdot}0429. \end{array} \tag{32}$$

The agreement is sufficient to justify the comparison, when we remember that the β-coefficients have only been determined to the fourth stage, and further, that the Stokes' series (29) for the C-coefficients presumably converge slowly in this extreme case.

It should be remarked that we do not gain information from this comparison about the convergence of the Stokes' series for the separate coefficients for higher values of the parameter; the result concerns the series for the elevation. We find that Stokes' series for the elevation becomes divergent at the crests when the parameter has the value 0·291..., so far as the numerical calculation has been carried.

In this connection reference may be made to Wilton,* who concluded that the Stokes' series certainly diverge for a parameter greater than $1/e$, and who estimated the limiting value to be in the neighbourhood of $\frac{1}{3}$.

Wilton works out in detail a numerical example with the parameter $b = 0{\cdot}316$, for comparison with the highest wave. According to the present analysis, this is beyond the limiting value for b; the series should be divergent at the crest. This may well be the case, notwithstanding that the coefficients C, as calculated by Wilton, diminish steadily as far as the order shown; since the series is supposed to be divergent only at the crests, one might expect the divergence to become evident numerically only after calculating a large number of terms. The example may serve as an illustration of Prof. Burnside's criticism, that it is necessary to know the limiting value of b before Stokes' series can be used with confidence for numerical calculation.

8. We may examine briefly the present method for waves of small height. It is of interest first to consider the exact expression

$$\frac{dw}{dz} = 2^{-1/3}(1-e^{2iw})^{1/3}. \tag{33}$$

We can integrate dz/dw and so obtain the equation of the stream-lines in finite form, and also exact expressions for the variations of pressure along any stream-line. To find how far (33) satisfies the condition for a free wave under constant pressure at a stream line $\psi = \alpha$, it is simpler to expand first before integrating; we can then express q^2 and y as cosine series. In this way we find at the wave surface $\psi = \alpha$, writing down the variable part only,

$$\begin{aligned}\text{Const.}\times(q^2-2gy) = {} & \{\tfrac{1}{3}ge^{-2\alpha}-(\tfrac{1}{3}e^{-2\alpha}-\tfrac{1}{27}e^{-6\alpha}-\tfrac{5}{729}e^{-8\alpha}-\ldots)\}\cos 2\phi \\ & +\{\tfrac{1}{9}ge^{-4\alpha}-(\tfrac{1}{9}e^{-4\alpha}-\tfrac{5}{243}e^{-8\alpha}-\ldots)\}\cos 4\phi \\ & +\{\tfrac{14}{243}ge^{-6\alpha}-(\tfrac{5}{81}e^{-6\alpha}-\tfrac{10}{729}e^{-10\alpha}-\ldots)\}\cos 6\phi \\ & +\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned} \tag{34}$$

Hence, if we take $g^{-1} = 1+\frac{1}{9}e^{-4\alpha}$, the pressure is constant up to, and including, terms in $e^{-4\alpha}$; and the next term is the small quantity $-\frac{1}{243}e^{-6\alpha}\cos 6\phi$. This value for $1/g$ is Stokes' expression $1+b^2$, the

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[*] J. R. Wilton, *loc cit. ante.*

parameter b having the value $\frac{1}{3}e^{-2\alpha}$ to this order. It is, of course, impossible to make the right-hand side of (34) zero for all values of ϕ merely by choosing g a suitable function of α. However, the fact that (33) satisfies approximately, to the order shown above, the conditions for a free wave of any height, explains the smallness of the coefficients β_1, β_2,... even for the highest wave when α is zero.

Returning to the general equations (15), for a numerical example of a wave of moderate height we take $e^{-2\alpha} = 3/10$. In this case we shall only carry the approximation to the second stage, to illustrate the character of the coefficients. We have the following numerical values:

$$B_{01} = 1{\cdot}008\,; \qquad B_{11} = -8{\cdot}24\,; \qquad B_{21} = -7{\cdot}336\,;$$

$$B_{03} = -0{\cdot}002\,; \qquad B_{13} = 2{\cdot}056\,; \qquad B_{23} = -8{\cdot}3\,;$$

$$B_{05} = -0{\cdot}004\,; \qquad B_{15} = 0{\cdot}082\,; \qquad B_{25} = 2{\cdot}1.$$

From these we obtain $\beta_1 = -0{\cdot}0018$; $\beta_2 = -0{\cdot}00066$. Making the comparison with a Stokes' series, as in the previous sections, we find

$$A_1 = -C_1 = 0{\cdot}1018, \qquad \tfrac{1}{2}A_2 = C_2 = 0{\cdot}0104, \qquad \tfrac{1}{3}A_3 = -C_3 = 0{\cdot}0016.$$

The numerical values confirm the impression that while the β-coefficients diminish indefinitely as the height of the wave becomes smaller, it is more difficult to obtain their values by the method of successive approximation used in dealing with the infinite set of equations for them.

The behaviour of the β-coefficients is made clearer by studying the leading terms B_{01}, B_{03}, ..., on the right-hand side of equations (15). Over the whole range for α, from zero to infinity, B_{01} only varies from about 1·24 to 1; consequently, from the first equation, g is never much different from unity. From the remaining equations, we see that β_1, β_2, β_3, ..., form a parallel series to B_{03}, B_{05}, B_{07}, ..., taken in order.

It is only for the highest wave ($\alpha = 0$) that all the terms of the latter series are positive and decrease steadily to zero from the first term; for other values, the series is not quite so simple in form, although in all cases the terms converge ultimately to zero. The character of these terms is best illustrated by numerical examples, such as are given in the following Table:

$e^{-2\alpha}$.	B_{01}.	B_{03}.	B_{05}.	B_{07}.	B_{09}.
1·0	1·2405	0·124	0·0443	0·0225	0·0136
0·9	1·111	0·083	−0·007	−0·013	−0·010
0·75	1·0806	−0·0167	−0·0157	−0·0126	−0·0071
0·3	1·008	−0·0022	−0·0042	−0·0013	−0·0004

The quantities $B_{0,2n+1}$, and, in fact, all the coefficients $B_{r,2n+1}$, can be studied algebraically from the relations (14). The algebraic solution of equations (15), together with a formal study of convergence, would be of great interest; meantime the numerical illustrations given in the foregoing discussion may serve to show the possibility of a general scheme which includes waves of any permissible height.

Experiments on the Effect of the Vibration of a Stretched Wire forming Part of a Closed Electric Circuit.

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In connection with some experiments involving the use of a thermo-microphone of small dimensions, I tried one of large proportions, to compare its efficiency with that of the small one. It consisted of a loop of silver gilt wire, 4 feet in length, stretched lightly between two glass insulators mounted on a stiff wooden batten. It was joined up in a circuit with a 10-volt battery and the primary winding of the step-up transformer of a three-valve (low frequency) amplifier, the secondary winding of the third valve of the amplifier leading to a pair of 60-ohm Brown telephones, which were used for recording the effects of sound vibrations on the loop of wire.

The listener at the telephones was situated in a quiet place some distance from the room in which this wire was fixed, and he could not hear with his naked ears the sounds produced to test the microphonic capabilities of the warmed wire. The response to these sounds in the telephones was well marked, the wire evidently acting as an efficient, though weak, microphone. Musical notes, produced by blowing organ pipes and a small syren, were loud and clear; laughing, whistling, and humming were easily differentiated, though speech was not intelligible.

The current flowing through the wire was $\frac{1}{4}$ ampère, and maintained its temperature slightly above that of the surrounding atmosphere. The voltage was then reduced in steps, and, though the intensity of the sound in the telephones also reduced, after a certain point, some sounds were still distinguishable after the battery had been disconnected and the circuit closed. The wire was now at the same temperature as that of the air.